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1985 J. Phys. A: Math. Gen. 18 1173

(http://iopscience.iop.org/0305-4470/18/8/018)

J. Phys. A: Math. Gen. 18 (1985) 1173-1192. Printed in Great Britain

Covariant derivation of the classical rotational dynamics of an extended charge. Analysis of a non-relativistic model

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Received 19 July 1984, in final form 14 November 1984

Abstract. In this work we present a covariant relativistic derivation of the equation of motion for a spinning extended charge. From it we study in the non-relativistic limit the rotational dynamics of a rigid spherical charge, comparing our results with previous ones. In particular some properties of this model concerning runaway and self-oscillating solutions are discussed. We show that, unlike for translational motion, there are no runaway solutions. We also analyse the rotational motion in presence of an external field.

1. Introduction

The study of the dynamics of charged particles with structure has revived the interest of physicists at different times since the end of the last century (Abraham 1903, Herglotz 1903, Lorentz 1956, Schott 1912, 1936, 1937) up to the present day (Bohm and Weinstein 1948, de la Peña *et al* 1982, França *et al* 1978, Jiménez and Montemayor 1983a, Markov 1946). (Excellent reviews of the equation of motion for charged particles can be found in Erber (1961) and Caldirola (1979).) However, these studies have been centred mainly on the analysis of the translational motion, whilst works about the rotational motion are rather scarce and in many cases they only deal with particular situations (for instance, the non-relativisitc limit (Jiménez and Montemayor 1983b), constantly directed angular velocity (Daboul and Jensen 1973, Daboul 1975, etc.), having obtained contradictory results (Jiménez and Montemayor 1983a, b, Rañada and Vázquez 1982, 1984). In all these works a covariant derivation of the equations of motion is missing. The importance of this becomes clear when analysing the translational case, in which the correct definition of the four-linear momentum does not coincide with the one to which a purely non-relativistic analysis would lead (Rohrlich 1965, 1970, Jackson 1975). Due to all this, the main purpose of this work is to give a correctly covariant relativistic derivation of the equation of rotational motion for an extended charge.

Another important point is the study of the properties of the rotational motion of the charge. Nevertheless, this crucially depends on the model we use for the structure of the charge. Most of the studies done up to now, both in the translational and the rotational case, employ a rigid spherically symmetric charge distribution. Obviously this occurs in a unique frame of reference. The Lorentz transformation enables us to

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know the structure in another inertial frame, where it is no longer spherical. A detailed study of the rigid charge has been carried out by Nodvik (1964). However, the problem is not over because the rigidity contradicts the principle of relativity of finite signal transmission velocity. Therefore, the study of a rigid charge can only be coherently made in the non-relativistic limit.

Our second aim is to obtain the non-relativistic limit of the equation of motion for a rigid spherically symmetric charge, and to study a few properties of.the motion in order to compare them with previous results (Daboul and Jensen 1973, Daboul 1975, Jiménez and Montemayor 1983b, Rañada and Vázquez 1982, 1984).

The outline of the paper is as follows. In § 2 we emphasise the necessity of a covariant derivation of the equation of motion for a spinning extended charge, that is obtained in § 3. In § 4 we consider a rigid spherical charge in the non-relativistic limit. Some properties of this model concerning runaway and self-oscillating solutions are discussed in § 5. Section 6 is devoted to the study of the rotational motion of a non-relativistic charge in presence of an external field. Section 7 contains a summary and discussion of the main results of the paper. We also include three appendices to obtain some results that are used in the main text.

2. Necessity of a covariant derivation

The derivations of the equation of rotational motion that can be found in Daboul and Jensen (1973), Daboul (1975) and Rañada and Vázquez (1982, 1984) rest essentially upon two assumptions. On the one hand, taking into account that both works consider a rigid spherical charge, they identify the intrinsic angular momentum $\boldsymbol{J}$ (spin) with $I\boldsymbol{\omega}$, $\boldsymbol{\omega}$ being the angular velocity and I the moment of inertia of the mass distribution. On the other hand, they consider that the variation of the angular momentum J with time is due, besides the external forces, to the Lorentz force that the self-field of the charge produces on itself. Although this seems trivially correct, the study of the translational motion shows us that it is not necessarily so. An analysis of this point can be found in Rohrlich (1965, 1970) and Jackson (1975), which also indicate the reason for the mistake and its solution. In order to know how to undertake correctly the rotational motion we sketch in the following the problem that arises in the translational case as well as its solution.

The situation is similar to the rotational motion. The difference is that now we are interested in the equation for the linear momentum $\boldsymbol{p}=m\gamma\boldsymbol{v}$ instead of the angular momentum. In the line of thought indicated above, the starting point would be the equation

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=\boldsymbol{F}_{\mathrm{ext}}+\int_{\mathbb{R}^3}\left(\rho\boldsymbol{E}_{(\mathrm{s})}+\frac{\boldsymbol{j}}{c}\times\boldsymbol{B}_{(\mathrm{s})}\right)\mathrm{d}^3r, \tag{1}$$

where $\boldsymbol{E}_{(\mathrm{s})}$ and $\boldsymbol{B}_{(\mathrm{s})}$ are the self-fields of the charge. (Remark that in Jackson (1975) no mechanical mass is considered, that is, $m\equiv 0$.)

With the aid of the Maxwell equations, and assuming that the acceleration goes to zero when $t\to-\infty$, it is easy to see that the self-force of equation (1) may be written as

$$\int_{\mathbb{R}^3}\left(\rho\boldsymbol{E}_{(\mathrm{s})}+\frac{\boldsymbol{j}}{c}\times\boldsymbol{B}_{(\mathrm{s})}\right)\mathrm{d}^3r=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{(\boldsymbol{E}_{(\mathrm{s})}\times\boldsymbol{B}_{(\mathrm{s})})}{4\pi c}\,\mathrm{d}^3r, \tag{2}$$

whence equation (1) becomes, using the proper time, $\mathrm{d}\tau = \gamma^{-1}\,\mathrm{d}t$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\boldsymbol{p}+\frac{1}{4\pi c}\int_{\mathbb{R}^3}\mathrm{d}^3r\,(\boldsymbol{E}_{(\mathrm{s})}\times\boldsymbol{B}_{(\mathrm{s})})\right)=\gamma\boldsymbol{F}_{\mathrm{ext}}. \tag{3}$$

The meaning of this equation is that the external forces produce the variation of the total momentum of the charge that includes the mechanical momentum $\boldsymbol{p}$ and the electromagnetic momentum $\boldsymbol{P}_{\mathrm{elm}}$, which would be

$$\boldsymbol{P}_{\mathrm{elm}}=(c/4\pi)\int\mathrm{d}^3r(\boldsymbol{E}_{(\mathrm{s})}\times\boldsymbol{B}_{(\mathrm{s})}). \tag{4}$$

The problem arises that $\boldsymbol{p}$ and $\gamma\boldsymbol{F}_{\mathrm{ext}}$ constitute the space components of four-vectors, whilst the quantity defined in equation (4) does not. It is then necessary to define in a covariant way the electromagnetic momentum of the charge. The correct definition, which can be found in Rohrlich (1965, 1970) and Jackson (1975), leads to modifying equation (3). For the sake of brevity, we refer the reader interested in the details to Rohrlich (1965) and Jackson (1975). The non-relativistic approach of the equation that is obtained when the momentum of the electromagnetic field is correctly defined can be found in de la Peña *et al* (1982).

The defect pointed out appears for instance in Bohm and Weinstein (1948). These authors, who only deal with the non-relativistic equation, start from equation (1). This leads them to an erroneous final counting of the mass of the charge, which has important consequences in the behaviour of the trajectories. In the exactly relativistic equation the mistake would be even more important.

What we have exposed makes doubtful the possibility of writing, for the rotational motion,

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}(I\boldsymbol{\omega})=\int\left[\rho\boldsymbol{r}\times\boldsymbol{E}_{(\mathrm{s})}+\boldsymbol{r}\times\left(\frac{\boldsymbol{j}}{c}\times\boldsymbol{B}_{(\mathrm{s})}\right)\right]\mathrm{d}^3r+\boldsymbol{M}_{\mathrm{ext}}, \tag{5}$$

as is done in Rañada and Vázquez (1982, 1984).

Using this line of thought, and with the analysis we have exposed as a basis, Jiménez and Montemayor (1983b) have looked for the correct equation introducing a definition of the momentum of each elementary volume of the mass distribution according to the remarks made above. However, this way of undertaking the problem is not very rigorous, which has impelled us to obtain the covariant derivation. It is clear that different results are obtained by Jiménez and Montemayor (1983b) and Rañada and Vázquez (1982, 1984). Only the covariant derivation will allow us to elucidate which of these is the right one, and the limits of validity of each equation.

3. Equation of motion for the rotation of a free extended charge

Our system consists of mass and charge distributions in the presence of their own electromagnetic field. The stability of the electron requires the existence of a field of cohesive forces in order to compensate for the mutual repulsion.

The study of these forces, first outlined by Poincaré (see Miller 1973), lies outside the scope of this paper. However, it is reasonable to assume that they do not contribute to the global motion of the charge, that is, both the net cohesive force over the whole charge and its net torque about any point are zero. Now, let $T_{\mathrm{m}}^{\mu\nu}$, $T_{\mathrm{elm}}^{\mu\nu}$ and $T_{\mathrm{c}}^{\mu\nu}$ be the energy-momentum tensor corresponding, respectively, to the mass distribution, the electromagnetic self-field and the cohesive forces. Convervation of energy and momentum demands the following relation to be fulfilled:

$$\partial_\mu (T^{\mu\nu}_{\mathrm{m}} + T^{\mu\nu}_{\mathrm{elm}} + T^{\mu\nu}_{\mathrm{c}}) = 0. \tag{6}$$

In order to study the global rotation of the charge we have to give a covariant definition of the angular momentum. Let $z^\mu(\tau)$ be the world line of one fixed point 0 of the charge distribution. If $v^\mu = \dot{z}^\mu$ denotes its four-velocity, we introduce the hypersurface $\Sigma(\tau)$ defined by the relation

$$(x^\mu - z^\mu(\tau))v_\mu(\tau) = 0. \tag{7}$$

It represents the three-space (t = constant) in the rest frame $S^{(0)}$ of the point 0. Let $\mathrm{d}\sigma^\mu = (v^\mu/c)\,\mathrm{d}\sigma$ be the element of hypersurface of $\Sigma(\tau)$, where $\mathrm{d}\sigma$ is Lorent invariant and coincides with the element of volume of the three-space in the frame $S^{(0)}$,

$$\mathrm{d}\sigma = -\frac{v_\mu}{c}\,\mathrm{d}\sigma^\mu = -\frac{v_\mu^{(0)}}{c}\,\mathrm{d}\sigma^{\mu(0)} = \mathrm{d}\sigma^{0(0)} = \mathrm{d}^3 r^{(0)}. \tag{8}$$

Now we define the angular momentum $J^{\mu\nu}_{\mathrm{m}}$ as

$$J^{\mu\nu}_{\mathrm{m}} = \int (-c^{-1})(T^{\alpha\mu}_{\mathrm{m}} x^\nu - T^{\alpha\nu}_{\mathrm{m}} x^\mu)\,\mathrm{d}\sigma_\alpha. \tag{9}$$

It is easy to see that, in the non-relativistic limit, by using the relation†

$$J_k = \tfrac{1}{2}\varepsilon_{kij} J^{ij}_{\mathrm{m}}, \tag{10}$$

we obtain the usual angular momentum

$$\boldsymbol{J} = \int \mathrm{d}^3 r\, \mu(r)(\boldsymbol{r} \times \boldsymbol{v}), \tag{11}$$

where $\mu(r)$ stands for the mass density.

To see that, we introduce in equation (9) the energy-momentum tensor of the mass distribution

$$T^{\mu\nu}_{\mathrm{m}} = -\mu(x)v^\mu(x)v^\nu(x), \tag{12}$$

where $\mu(x)$ denotes the proper mass distribution, that is the mass distribution of the point x in its rest frame, and $v^\mu(x)$ its four-velocity:

$$J^{\mu\nu}_{\mathrm{m}} = c^{-2} \int \mu(x)[v^\alpha(x)v^\mu(x)x^\nu - v^\alpha(x)v^\nu(x)x^\mu]v_\alpha(\tau)\,\mathrm{d}\sigma. \tag{13}$$

Specialising to the rest frame of the point 0, $v^\mu(\tau) = (c, \boldsymbol{0})$, whence equation (13) becomes

$$J^{\mu\nu(0)}_{\mathrm{m}} = -c^{-1} \int \mu(t, \boldsymbol{r}^{(0)})(v^\mu x^{\nu(0)} - v^\nu x^{\mu(0)})v^0\,\mathrm{d}^3 r^{(0)}. \tag{14}$$

In the non-relativistic limit we have

$$v^\mu(x) = (c\gamma, \boldsymbol{v}\gamma) \sim (c, \boldsymbol{v}),$$

† Here, Latin indices run over the space components.

and the angular momentum

$$J_k = -\tfrac{1}{2}\varepsilon_{kij}\int \mu(t, \boldsymbol{r}^{(0)})(v^i x^{j(0)} - v^j x^{i(0)})\,\mathrm{d}^3 r^{(0)} = \int \mu(t, \boldsymbol{r})\varepsilon_{kij}(x^i v^j)\,\mathrm{d}^3 r,$$

results, which coincides with the expression (11).

Because we are going to study the rotational motion, we suppose that the external forces do not produce variation of the net momentum of the charge, and then there is a point of the distribution whose velocity remains unchanged†. Then we choose this point as the one used to define the hypersurface $\Sigma(\tau)$, that is the point 0. We also work, for the sake of simplicity, in the frame in which the space components of $z^\mu(\tau)$ coincide with the origin

$$z^\mu(\tau) \equiv (ct, \boldsymbol{0}) \equiv (c\tau, \boldsymbol{0}) \tag{15}$$

whence the proper time of 0 is the coordinate time of our frame, and $\Sigma(\tau)$ coincides with the three-space. Then

$$\mathrm{d}\sigma^\mu \equiv (1, \boldsymbol{0})\,\mathrm{d}^3 r, \tag{16}$$

and equation (9) becomes

$$J_{\mathrm{m}}^{\mu\nu}(t) = c^{-1}\int \mathrm{d}^3 r[T_{\mathrm{m}}^{0\mu}(\boldsymbol{r}, t)x^\nu - T_{\mathrm{m}}^{0\nu}(\boldsymbol{r}, t)x^\mu]. \tag{17}$$

In order to obtain the equation of motion, we derive equation (17) with respect to time, and using equation (6) we get

$$\dot{J}_{\mathrm{m}}^{\mu\nu}(t) = -\int \mathrm{d}^3 r\left(\frac{\partial T_{\mathrm{elm}}^{\lambda\mu}}{\partial x^\lambda}x^\nu - \frac{\partial T_{\mathrm{elm}}^{\lambda\nu}}{\partial x^\lambda}x^\mu\right), \tag{18}$$

where we have supposed, according to our assumption, that the net torque due to the cohesive forces is zero.

This equation can also be derived from a general relation obtained by Kaup (1966) within the context of special relativity, specialising another one by Dixon (1964) valid in the general theory. Following our notation, Kaup's relation can be written as follows. Let $\Lambda_{\beta\ldots}^{\alpha\ldots\mu}$ denote some arbitrary tensor of any rank, depending on both x^μ and $z^\mu(\tau)$. Here, x^μ is an arbitrary point on the hypersurface $\Sigma(\tau)$. The tensor $\Lambda_{\beta\ldots}^{\alpha\ldots\mu}$ fulfils the relation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Lambda_{\beta\ldots}^{\alpha\ldots} &\equiv \frac{\mathrm{d}}{\mathrm{d}s}\int \Lambda_{\beta\ldots}^{\alpha\ldots\mu}\,\mathrm{d}\sigma_\mu \\ &= \int \frac{\partial\Lambda_{\beta\ldots}^{\alpha\ldots\mu}}{\partial x^\mu}\left(1 - \frac{(x^\nu - z^\nu)a_\nu}{c^2}\right)\frac{v^\rho}{c}\,\mathrm{d}\sigma_\rho + \int \frac{v^\lambda}{c}\frac{\partial\Lambda_{\beta\ldots}^{\alpha\ldots\mu}}{\partial z^\lambda}\,\mathrm{d}\sigma_\mu,\end{aligned} \tag{19}$$

where $a^\alpha = \mathrm{d}v^\alpha/\mathrm{d}\tau$ and $\mathrm{d}s/c \equiv \mathrm{d}\tau$ is the proper time.

In our case, for the frame of reference we have chosen $\tau \equiv t$, $z^\mu(\tau) \equiv (ct, \boldsymbol{0})$, $v^\mu/c \equiv (1, \boldsymbol{0})$, $a^\mu \equiv (0, \boldsymbol{0})$ and $\mathrm{d}\sigma^\mu = \mathrm{d}^3 r(1, \boldsymbol{0})$ whence equation (19) reads

$$\frac{\mathrm{d}\Lambda_{\beta\ldots}^{\alpha\ldots}}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \Lambda_{\beta\ldots}^{\alpha\ldots0}\,\mathrm{d}^3 r = -c\int \frac{\partial\Lambda_{\beta\ldots}^{\nu\ldots\mu}}{\partial x^\mu}\,\mathrm{d}^3 r - c\int \frac{\partial\Lambda_{\beta\ldots}^{\alpha\ldots0}}{\partial z^0}\,\mathrm{d}^3 r. \tag{20}$$

† Although this last affirmation seems obvious, it is not so trivial. We know that in certain approaches there are solutions that fulfil it. But even in these cases, other solutions that demand a different treatment are possible (Bohm and Weinstein 1948).

Specialising to the tensor

$$\Lambda_{\beta\ldots}^{\alpha\ldots\mu} \to J_{\mathrm{m}}^{\mu\alpha\sigma}(x) = -c^{-1}(T^{\mu\alpha}(x)x^{\sigma} - T^{\mu\sigma}(x)x^{\alpha}), \tag{21}$$

and using equation (6), we obtain equation (18).

The derivation we have made turns out to be more transparent in our case.

We return to equation (18) and introduce the relation

$$\partial_{\lambda} T_{\mathrm{elm}}^{\lambda\mu} = c^{-1} F_{(\mathrm{s})}^{\mu\nu} j_{\nu} \tag{22}$$

where j^{μ} stands for the four-current and $F_{(\mathrm{s})}^{\mu\nu}$ for the electromagnetic tensor of the self-field of the charge.

Writing now $j^{\mu} \equiv e\rho(t, \boldsymbol{r})c(1, \boldsymbol{v}/c)\gamma(v)$, where $\boldsymbol{v}$ is the three-velocity of an arbitrary point of the distribution, ρ the normalised (to unity) proper density of charge, and e the total charge, we get by using equations (10), (18) and (22)

$$\dot{\boldsymbol{J}} = e \int \mathrm{d}^3 r(\boldsymbol{r} \times (\boldsymbol{E}_{(\mathrm{s})} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}_{(\mathrm{s})}]\rho^*(t, \boldsymbol{r}), \tag{23}$$

where $\rho^* = \gamma\rho$ is the charge density in the frame of reference in which we are working.

This equation is completely exact. On the contrary, equation (5), as used in Rañada and Vázquez (1982, 1984), is only valid when $\rho^* \simeq \rho$, i.e. for small velocities. (Note that for spherical charges ρ, in contrast with ρ^*, is independent of t.) Moreover, the identification $\boldsymbol{J} = I\boldsymbol{\omega}$ loses its meaning in the relativistic case, in the sense that I should be velocity dependent.

It is to be noted that, unlike the translational motion, the self-force suffices to give an account of the motion. In the translational case an extra term appears in the covariant definition of the linear momentum. Based upon this result, Jiménez and Montemayor (1983b) studied the rotational motion, introducing a definition for the angular momentum that includes this extra term in the linear momentum of each element of the charge. We want to emphasise that the result we have obtained tells us that in the purely rotational motion such a procedure leads to a wrong equation. The non-existence of this extra term leads to different properties, especially as concerns the runaway solutions and the renormalisation of I, as we shall see in §§ 5 and 6.

4. The equation of motion for a rigid spherical non-relativistic charge

It is clear that a theory of extended charges requires the elaboration of a model about the structure of charge. However, in order to begin to obtain some insight about the behaviour of these systems, this problem can be avoided if a rigid spherically symmetric structure is considered. This contradicts the postulates of relativity, because a perturbation in one point of the distribution would produce an instantaneous effect over the whole charge. Such a phenomenon shows the necessity of a more elaborate model where all contradictions have been removed. However, if a non-relativistic analysis is made, the rigid model can be accepted as valid. Moreover it leads, without too many difficulties, to some results.

Let us see how expression (23) reads in this case. Firstly, the point $z^{\mu}(t)$ will be made to coincide with the geometric centre of the charge. Moreover $\gamma \sim 1$ and we may write $\rho^*(t, \boldsymbol{r}) \sim \rho(r)$. The self-field will be calculated by means of the retarded potentials

$$\boldsymbol{E}_{(s)} = -\nabla\phi_{(s)} - c^{-1}\,\partial\boldsymbol{A}_{(s)}/\partial t, \tag{24a}$$

$$\boldsymbol{B}_{(s)} = \nabla\times\boldsymbol{A}_{(s)}, \tag{24b}$$

$$\boldsymbol{A}_{(s)} = \frac{e}{c}\int \frac{\rho(r')\boldsymbol{v}(t_r, \boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3r', \tag{24c}$$

$$\phi_{(s)} = e\int \frac{\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3r', \tag{24d}$$

where t_r denotes the retarded time, $t_r = t - c^{-1}|\boldsymbol{r}-\boldsymbol{r}'|$. Note that the spherical symmetry makes ρ independent of time. For this the scalar potential does not include retardation effects. Let us consider separately the contributions of the self-electric field and the self-magnetic field

$$\boldsymbol{N}_E^{(s)} = e\int \mathrm{d}^3r(\boldsymbol{r}\times\boldsymbol{E}_{(s)})\rho(r), \tag{25a}$$

$$\boldsymbol{N}_B^{(s)} = (e/c)\int \mathrm{d}^3r[\boldsymbol{r}\times(\boldsymbol{v}\times\boldsymbol{B}_{(s)})]\rho(r). \tag{25b}$$

Taking into account that

$$\boldsymbol{v}(t, \boldsymbol{r}) = \boldsymbol{\omega}(t)\times\boldsymbol{r}, \tag{26}$$

where $\boldsymbol{\omega}(t)$ stands for the instantaneous angular velocity, we have, using equations (24) and (25),

$$\boldsymbol{N}_E^{(s)} = -\frac{e^2}{c^2}\int \mathrm{d}^3r\int \mathrm{d}^3r'\frac{\rho(r)\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{r}\times[\Gamma\boldsymbol{\omega}(t_r)\times\boldsymbol{r}'] \tag{27a}$$

$$\boldsymbol{N}_B^{(s)} = (e/c)\int \mathrm{d}^3r\rho(r)\boldsymbol{r}\cdot\boldsymbol{B}_{(s)}(\boldsymbol{r}, t)[\boldsymbol{\omega}(t)\times\boldsymbol{r}]. \tag{27b}$$

Note that, because of the spherical symmetry and the non-retarded character of ϕ, this term does not contribute to $\boldsymbol{N}_E^{(s)}$. Explicit calculation of the above quantities can be found in appendices 1 and 2. Here we only give the result

$$\boldsymbol{N}_E^{(s)} = -\tfrac{8}{3}\pi e^2\int_{-\infty}^{t} \mathrm{d}t'\; G(t-t')\boldsymbol{\omega}(t'), \tag{28a}$$

$$\boldsymbol{N}_B^{(s)} = \tfrac{8}{3}\pi e^2\boldsymbol{\omega}(t)\times\int_{-\infty}^{t} \mathrm{d}t'\; G(t-t')\dot{\boldsymbol{\omega}}(t'), \tag{28b}$$

where

$$G(t) = t\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)(\boldsymbol{r}\cdot\boldsymbol{R}+r^2)|_{R=ct} \tag{29a}$$

$$= \frac{4\pi}{c}\int_0^\infty k[\tilde{\rho}'(k)]^2\sin(kct)\,\mathrm{d}k, \tag{29b}$$

$$\tilde{\rho}(k) = (2\pi)^{-3/2}\int \mathrm{d}^3\mathrm{r}\,\rho(r)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{30}$$

The expression we have obtained for $\boldsymbol{N}_E^{(s)}$ corrects the one obtained by Jiménez and Montemayor (1983b).

Let us now express the angular momentum as a function of $\boldsymbol{\omega}$. From equations (11) and (26) the non-relativistic limit of $\boldsymbol{J}$ can be written as

$$\boldsymbol{J} = I_{\text{mec}}\boldsymbol{\omega} \tag{31}$$

where I_{mec} stands for the moment of inertia of the spherical mass distribution

$$I_{\text{mec}} = \int \mathrm{d}^3 r\, \mu(r) r^2. \tag{32}$$

Finally, from (23) and (28) the equation of motion may be written

$$I_{\text{mec}}\,\dot{\boldsymbol{\omega}} = -\tfrac{8}{3}\pi e^2 \int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')\dot{\boldsymbol{\omega}}(t') + \tfrac{8}{3}\pi e^2 \boldsymbol{\omega}(t) \times \int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')\boldsymbol{\omega}(t'). \tag{33}$$

This equation coincides with the one obtained by Rañada and Vazquez (1982, 1984). Nevertheless in these papers it appears as an exact relativistic equation, whilst our derivation shows, as has been pointed out at the end of § 3, that it is only valid for small velocities. Moreover the moment of inertia should be dependent on the angular velocity unless terms of order $(v/c)^2$ are neglected. Other problems arise from the rigidity condition. We do not enter into discussion about them because the inadequacy of the rigidity in a relativistic system is clear. Then we just emphasise that equation (33) is correct only up to terms v^2/c^2.

As concerns the second term of the RHS of (33), we are going to show that, if the angular acceleration does not change very quickly, it is of order v/c with respect to the first term. Let r_e be the radius of the electron, and $\tau_e = r_e/c$. The condition of non-relativistic velocities can be expressed, imposing that the velocity of a point separated from the origin of the charge a distance r_e is much smaller than c, that is

$$\omega\tau_e \equiv v_r/c \ll 1. \tag{34}$$

Moreover it is easy to see with the aid of equation (29) that the lifetime of the kernel $G(t)$ is of order $2\tau_e$.

If we now suppose that

$$|\ddot{\boldsymbol{\omega}}|\tau_e \ll |\dot{\boldsymbol{\omega}}| \tag{35}$$

the first term of the RHS of (33) can be estimated to be of order $|\dot{\boldsymbol{\omega}}|\eta$ where

$$\eta = \tfrac{8}{3}\pi e^2 \int_0^{\infty} G(t)\,\mathrm{d}t. \tag{36}$$

As concerns the second term, we can estimate it as follows. From condition (34) in the time interval in which G is not negligible, $\boldsymbol{\omega}$ has shifted at most by an angle

$$\Delta\varphi \sim |\Delta\omega|/|\omega| \sim \dot{\omega}\tau_e/\omega \tag{37}$$

whence the value of that term is of order

$$|\eta\omega^2 \sin\Delta\varphi| < \eta\omega^2\Delta\varphi. \tag{38}$$

The ratio of both terms is then of order

$$|(\eta\omega^2 \sin\Delta\varphi)/\eta\dot{\omega}| < \omega\tau_e \sim v_r/c \ll 1 \tag{39}$$

which is the result we were looking for.

However, it has to be noted that in general condition (35) is independent of the non-relativistic limit, and then some care has to be taken before removing the last term of (33). For the moment we maintain that entire expression.

5. Some properties of the non-relativistic model

In this section we only give a few first results concerning the non-trivial solutions of (33), particularly the runaway and self-oscillating solutions. In most works about this subject, the translational motion included, this problem has been simplified by reducing it to the study of the solutions of type

$$\boldsymbol{\omega}(t)=\boldsymbol{\omega}_0\,\mathrm{e}^{\nu t}. \tag{40}$$

However, one could think of much more general solutions with indefinitely increasing amplitudes and, maybe, time varying frequencies. In our case, one could even think of solutions whose direction would also change. We do not intend to solve these questions, but only to call attention to the fact that the study of solutions of type $\mathrm{e}^{\nu t}$, that we undertake in the following, constitutes only a partial aspect of the problem.

We consider two cases, in order to include both the runaway and the self-oscillating solutions: (a) Re $\nu>0$ and (b) Re $\nu=0$.

(a) Re $\nu>0$

Substituting (34) in (33) we have

$$\begin{aligned} I_{\mathrm{mec}}\boldsymbol{\omega}_0\nu\,\mathrm{e}^{\nu t} &= -\tfrac{8}{3}\pi e^2\int_{-\infty}^{t}\mathrm{d}t'\,G(t-t')\boldsymbol{\omega}_0\nu\exp(\nu t') \\ &= -\tfrac{8}{3}\pi e^2\int_0^{\infty}G(t')\boldsymbol{\omega}_0\nu\exp[\nu(t-t')]\,\mathrm{d}t', \end{aligned} \tag{41}$$

whence

$$I_{\mathrm{mec}}=-\tfrac{8}{3}\pi e^2\,\tilde{G}(\nu), \tag{42}$$

$\tilde{G}(\nu)$ being the Laplace transform of G.

As Re $\nu>0$, we may calculate explicitly $\tilde{G}(\nu)$ from (29b):

$$\tilde{G}(\nu)=4\pi\int_0^{\infty}\mathrm{d}k\,[\tilde{\rho}'(k)]^2\frac{k^2}{\nu^2+k^2c^2}. \tag{43}$$

From (42) two equations are obtained,

$$I_{\mathrm{mec}}=-\tfrac{8}{3}\pi e^2\,\mathrm{Re}\,\tilde{G}(\nu), \tag{44a}$$

$$I_{\mathrm{m}}\,\tilde{G}(\nu)=4\pi\int_0^{\infty}\mathrm{d}k\,[\tilde{\rho}'(k)]^2\frac{k^2(-2)\nu_{\mathrm{R}}\nu_{\mathrm{I}}}{(\nu_{\mathrm{R}}^2-\nu_{\mathrm{I}}^2+k^2c^2)^2+4\nu_{\mathrm{I}}^2\nu_{\mathrm{R}}^2}=0, \tag{44b}$$

where ν_{R} and ν_{I} are the real and imaginary parts of ν. Since $\nu_{\mathrm{R}}>0$, equation (44b) implies $\nu_{\mathrm{I}}=0$. Then Re $\tilde{G}(\nu)\equiv\tilde{G}(\nu)>0$, and (44$a$) has no solutions, because $I_{\mathrm{mec}}>0$. The conclusion is clear: no runaway solution of the form $\boldsymbol{\omega}_0\,\mathrm{e}^{\nu t}$ can be found.

(*b*) *Re* $\nu = 0$

We put $\nu = \mathrm{i}\lambda$. The calculation of $\tilde{G}(\mathrm{i}\lambda)$ must be modified, because

$$\int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}At} = \pi\delta(A) - \mathrm{i}\, V_{\mathrm{p}} A^{-1} \tag{45}$$

where V_{p} denotes the Cauchy principal value. Therefore, in this case we obtain

$$\tilde{G}(\mathrm{i}\lambda) = \frac{2\pi}{\mathrm{i}c^3}\lambda[\tilde{\rho}'(\lambda/c)]^2 - 4\pi V_{\mathrm{p}} \int_0^{\infty} \frac{k^2}{\lambda^2 - k^2c^2}\tilde{\rho}'(k)]^2\, \mathrm{d}k \tag{46}$$

whence (42) gives

$$I_{\mathrm{mec}} = \tfrac{32}{3}\pi^2 e^2 V_{\mathrm{p}} \int_0^{\infty} \frac{k^2}{\lambda^2 - k^2c^2}[\tilde{\rho}'(k)]^2\, \mathrm{d}k, \tag{47a}$$

$$0 = (2\pi/c^3)\lambda[\tilde{\rho}'(\lambda/c)]^2. \tag{47b}$$

It is trivial to see that $\lambda = 0$ is not a solution. Unlike case (a), equation (47*a*) could have solutions and then self-oscillations are possible.

The results we have obtained are similar to the ones for the translational motion but there are some important differences. As concerns the runaway solutions, instead of equation (44*a*) we obtain a similar one with the very important difference that the mass on the left of the equation (that appears in the translational motion instead of the moment of inertia) includes a renormalisation term which can make it take negative values. Consequently, the corresponding equation may have solutions. In our case, the renormalisation term does not exist and solutions are not possible. Concerning the self oscillations, the non-negativity of the moment of inertia does not prevent equation (47*a*) having solutions. Finally, in both cases, $\tilde{\rho}$ appears in the translational case, instead of $\tilde{\rho}'$.

6. Analysis of the non-relativistic charge in presence of an external field

If there is an external field acting on the charge, we may repeat the calculations of § 2. It suffices to replace the self-fields by the total fields, $\boldsymbol{E}_{(\mathrm{s})} + \boldsymbol{E}_{\mathrm{ext}}$ and $\boldsymbol{B}_{(\mathrm{s})} + \boldsymbol{B}_{\mathrm{ext}}$, in equation (22). The result is the inclusion in the RHS of (33) of a new term, $\boldsymbol{M}_{\mathrm{ext}}$, which can be written as

$$\boldsymbol{M}_{\mathrm{ext}} = e \int \mathrm{d}^3 r(\boldsymbol{r} \times \boldsymbol{E}_{\mathrm{ext}})\rho(r) + \frac{e}{c}\int \mathrm{d}^3 r[\boldsymbol{r} \times (\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{ext}})]\rho(r) \tag{48}$$

where we have assumed that the external field only produces rotation, that is, the net external force acting on the whole charge is zero.

In the following, we only analyse two cases: (a) $\boldsymbol{M}_{\mathrm{ext}} =$ constant, and (b) $\boldsymbol{E}_{\mathrm{ext}} = 0$ and $\boldsymbol{B}_{\mathrm{ext}} =$ constant.

(*a*) M^{ext} *constant. Renormalisation of I*

Here we are mainly interested in the occurrence of a phenomenon which is similar to the mass renormalisation appearing in the case of the purely translational motion. It is assumed that M_{ext} is small enough, so that the quadratic term in $\boldsymbol{\omega}$ of (33) can be neglected. Then this equation becomes

$$I_{\text{mec}}\dot{\boldsymbol{\omega}} = -\tfrac{8}{3}\pi e^2 \int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')\dot{\boldsymbol{\omega}}(t') + \boldsymbol{M}_{\text{ext}}, \tag{49}$$

the solution of which is trivially

$$\dot{\boldsymbol{\omega}} = \boldsymbol{M}_{\text{ext}}/I_{\text{r}} \tag{50}$$

where I_{r} is an effective or renormalised moment of inertia, and is given by

$$I_{\text{r}} = I_{\text{mec}} + \tfrac{8}{3}\pi e^2 \int_0^{\infty} G(t)\,\mathrm{d}t = I_{\text{mec}} + \eta. \tag{51}$$

Even if we consider the two terms on the RHS of (33), in the case that $\boldsymbol{\omega}(0)$ is parallel to $\boldsymbol{M}_{\text{ext}}$, the expression (50) is still the correct solution. Let us see now what is the meaning of the additional term η of (51).

Using (29a) it is easy to see that

$$\eta = \frac{2e^2}{3c^2}\int \mathrm{d}^3r\,\mathrm{d}^3r' \frac{\rho(r)\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{r}\cdot\boldsymbol{r}'. \tag{52}$$

After some algebra (see appendix 3), we can state the next relation

$$\mathscr{F} = \frac{e^2}{2c^2}\int \mathrm{d}^3r\,\mathrm{d}^3r' \frac{\boldsymbol{j}(\boldsymbol{r},t)\cdot\boldsymbol{j}(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|} = \tfrac{1}{2}\eta\omega^2(t), \tag{53}$$

where

$$\boldsymbol{j}(\boldsymbol{r},t) = \rho(r)[\boldsymbol{\omega}(t)\times\boldsymbol{r}(t)]. \tag{54}$$

The meaning of $\mathscr{F}$ remains clear in the case where $\boldsymbol{\omega}$ = constant: $\mathscr{F}$ is the magnetic energy of self-interaction of the charge. The renormalised energy of rotation can be written as

$$\mathscr{E}_{\text{rot}} = \tfrac{1}{2}I_{\text{r}}\omega^2 = \tfrac{1}{2}I_{\text{mec}}\omega^2 + \tfrac{1}{2}\eta\omega^2 = \tfrac{1}{2}I_{\text{mec}}\omega^2 + \mathscr{F}. \tag{55}$$

We see the resemblance with the translational case, in which the electrostatic self-energy has to be added to the purely mechanical one, in order to account for the actually observed mass (França *et al* 1978, de la Peña *et al* 1982, Rohrlich 1965, Alvarez-Estrada and Ros Martinez 1981). However, unlike the translational case, if we apply a torque at time $t=0$ to a particle at constant angular velocity ($\dot{\omega}(t')=0$, $t'<0$), it is easy to see from equation (49) that the inertia immediately after $t=0$ has a purely mechanical origin. Since $I_{\text{mec}}>0$, this is the essential reason for the non-existence of runaway solutions. On the contrary, in the translational case the inertia immediately after the application of an external force includes a part of the electromagnetic mass (de la Peña *et al* 1982), in such a way that it can take negative values and runaway solutions are possible.

(b) *Constant magnetic field*

In presence of a constant magnetic field, the external torque can be written

$$\boldsymbol{M}_{\text{ext}} = \Lambda\boldsymbol{\omega}\times\boldsymbol{B} \tag{56}$$

where

$$\Lambda \boldsymbol{\omega} = (2c)^{-1} \int \mathrm{d}^3 r\, (\boldsymbol{r} \times \boldsymbol{j}) = (3c)^{-1} \int \mathrm{d}^3 r\, r^2 \rho(r) \boldsymbol{\omega} \tag{57}$$

stands for the magnetic moment of the charge.

Similarly to the former case, we assume that $\boldsymbol{B}$ is not too large in order to neglect the quadratic term in $\boldsymbol{\omega}$ of equation (33). Then in this case the equation of motion is

$$I_{\mathrm{mec}} \dot{\boldsymbol{\omega}} = -\tfrac{8}{3}\pi e^2 \int_{t_0}^{t} \mathrm{d}t'\, G(t-t') \dot{\boldsymbol{\omega}}(t') + \Lambda \boldsymbol{\omega} \times \boldsymbol{B} \qquad t > t_0, \tag{58}$$

where we assume that the field $\boldsymbol{B}$ is switched on at $t = t_0$, and that for $t < t_0$, $\dot{\boldsymbol{\omega}} \equiv 0$.

Considering now three right-handed orthonormal vectors, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{B}/|\boldsymbol{B}|$, expanding $\boldsymbol{\omega}$ in this basis,

$$\boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_{\|}),$$

and defining the complex quantity

$$\Omega = \omega_1 + \mathrm{i}\omega_2, \tag{59}$$

the three equations (58) can be separated into two parts

$$I_{\mathrm{mec}} \dot{\omega}_{\|} = -\tfrac{8}{3}\pi e^2 \int_{t_0}^{t} \mathrm{d}t'\, G(t-t') \dot{\omega}_{\|}(t'), \qquad t > t_0, \tag{60a}$$

$$I_{\mathrm{mec}} \dot{\Omega} = -\tfrac{8}{3}\pi e^2 \int_{t_0}^{t} \mathrm{d}t'\, G(t-t') \dot{\Omega}(t') - \Lambda B \mathrm{i} \Omega(t), \qquad t > t_0. \tag{60b}$$

The first of these equations admits only the trivial solution $\omega_{\|} = \text{constant}$ (Petrovski 1957). As concerns the second one, we can use the Laplace transform method to obtain the solutions, putting $t_0 = 0$ for the sake of simplicity. However, our main goal is the knowledge of the stationary regime of the solutions. For this, it is preferable to set $t_0 = -\infty$ and to use the Fourier transform.

Introducing the new kernel

$$G^{(0)}(t) = G(t)\theta(t), \tag{61}$$

θ being the Heaviside jump function, the equation of motion for $\Omega(t)$ can be written

$$I_{\mathrm{mec}} \dot{\Omega} = -\tfrac{8}{3}\pi e^2 \int_{-\infty}^{\infty} G^{(0)}(t-t') \dot{\Omega}(t')\, \mathrm{d}t' - \Lambda B \mathrm{i} \Omega(t), \tag{62}$$

and its Fourier transform

$$-\mathrm{i}\lambda I_{\mathrm{mec}} \bar{\Omega}(\lambda) = -\tfrac{8}{3}\pi\, e^2 \overline{G^{(0)}}(\lambda)(-\mathrm{i}\lambda)\bar{\Omega}(\lambda) - \Lambda B \mathrm{i} \bar{\Omega}(\lambda), \tag{63}$$

with

$$\bar{f}(\lambda) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{d}t. \tag{64}$$

Then

$$\bar{\Omega}(\lambda)[-\mathrm{i}\lambda(I_{\mathrm{mec}} + \tfrac{8}{3}\pi e^2 \overline{G^{(0)}}(\lambda)) + \Lambda B \mathrm{i}] = 0, \tag{65}$$

whence if follows that

$$\bar{\Omega}(\lambda)=\sum_r \Omega_r \delta^{(K_r-1)}(\lambda-\lambda_r) \tag{66}$$

and

$$\Omega(t)=(2\pi)^{-1}\sum_r \Omega_r \exp(-\mathrm{i}\lambda_r t)(\mathrm{i}t)^{k_r-1} \tag{67}$$

where the λ_r's are the real zeros of the coefficient of $\bar{\Omega}(\lambda)$ in (65), and k_r their multiplicities.

Now, taking into account (61) and (64), we obtain

$$\overline{G^{(0)}}(\lambda)=\tilde{G}(-\mathrm{i}\lambda) \tag{68}$$

whence the λ_r's fulfil the following equations:

$$-\lambda_r\left(I_{\mathrm{mec}}+\tfrac{32}{3}\pi^2 e^2\int_0^\infty \frac{k^2}{k^2c^2-\lambda_r^2}[\tilde{\rho}'(k)]^2\,\mathrm{d}k\right)+\Lambda B=0, \tag{69a}$$

$$\lambda_r[\tilde{\rho}'(\lambda_r/c)]^2=0. \tag{69b}$$

Again $\lambda=0$ is not a solution. Then the actual frequencies λ_r are those ones that fulfil (69a) and

$$\tilde{\rho}'(\lambda_r/c)=0. \tag{69b'}$$

In consequence, generally speaking, there will be solutions for only particular values of B. In the other cases $\Omega=0$ for $t\to\infty$.

To sum up, for the total solution of equation (58), we can say that, in general, after the transient, it has the form $\boldsymbol{\omega}=\boldsymbol{\omega}_\parallel=$ constant, except for some special values of B for which a precession of the angular velocity $\boldsymbol{\omega}$ about the direction of $\boldsymbol{B}$ can exist, depending on whether or not $\tilde{\rho}'$ vanishes. That is, in general, the angular velocity $\boldsymbol{\omega}$ aligns itself with the magnetic field.

Furthermore $\boldsymbol{\omega}=\boldsymbol{\omega}_\parallel=$ constant remains the stationary solution if we include in the equation of motion the quadratic term in $\boldsymbol{\omega}$ of (33), unless precession occurs. In this case the transient behaviour should be studied. It is to be hoped that the unique form for the stationary regime of the solution will be $\boldsymbol{\omega}=\boldsymbol{\omega}_\parallel=$ constant.

7. Conclusions and discussion

The fundamental aspect of the present work is the attainment of the equation of rotational motion as a relation between tensor quantities, which guarantees its Lorentz invariant character. This allows us to obtain the correct form of the non-relativistic limit for that motion. With respect to previous work about the rotation of extended charges, the results we have obtained enable us to draw the following conclusion, namely, that equation (33), which is presented in Daboul and Jensen (1973), Daboul (1975) and Rañada and Vazquez (1982, 1984) as though it was exact, appears to be valid only in the low-velocity limit.

The essential reason for this is, besides the incoherence of dealing with a rigid structure, that the charge density appearing in equation (23), ρ^*, is velocity dependent, whilst equation (33) is only valid if one neglects this dependence, which can be done, in general, only if $v \ll c$.

We have also studied some properties of the motion of the charge both in absence of external fields and in presence of constant external torque and constant magnetic field.

We have found that in the free case, self-oscillations may or may not exist depending on the shape of the charge distribution. In any case, it is hoped that such solutions will decrease when an exact equation is considered. In principle we should agree with Daboul and Jensen (1973) and Daboul (1975), in the sense that the charge is expected to radiate indefinitely until the lowest energy state is attempted. However, it is well known that non-radiating confined (see e.g. Devaney and Wolf 1973) and unconfined (Fargue 1981) current distributions exist in the context of relativity theory. Then the question arises whether or not a coherent more refined model of extended charge will admit exactly non-radiating solutions.

In the presence of a constant magnetic field, the vector aligns itself with the field, the value of the parallel component remaining unchanged; the evolution is such that the normal component vanishes. Exceptions to this behaviour are found, in which the angular velocity precesses about B if the field strength takes some particular values.

The features we have found in the rotational motion are very similar to the ones that appear in the translational motion. However, some differences are found. For instance, translational self-oscillations appear in frequencies that cancel the corresponding Fourier component of the charge density whilst the rotational ones cancel the derivative of the Fourier components.

The most important difference emerges in the question of the runaway solutions. Translational motion does have these, contrary to rotational motion that does not. The reason lies in the fact that the initial inertia is different in the two cases. In the translational case the existence of runaway solutions depends on the sign of this inertia that includes part of the renormalisation, whereas in the rotational motion the equivalent magnitude, the mechanical moment of inertia, is always positive. Further investigation is needed to analyse the physical reasons for this difference.

Another question arises in connection with the solution of equation (33). It is obvious that $\boldsymbol{\omega} = \text{constant}$ satisfies it. However, it represents a motion where all points of the charge are continuously accelerated, and then, in general, radiation exists. Therefore the charge should tend to stop the motion, which is not the case. One could think that releasing rigidity and low-velocity conditions would eliminate $\boldsymbol{\omega} = \text{constant}$ as a solution. However, it is easy to see that this is not true. If one considers, as is physically reasonable, that a non-rigid charge would have, if rotating at constant angular velocity, cylindrical symmetry, it is easy to see that again $\boldsymbol{\omega} = \text{constant}$ is a relativistically exact solution. An explanation is found because in that case no quantity depends on t and the fields the charge creates outside its domain go as r^{-2} when r tends to infinity. Then the charge does not radiate energy, contrary to what one would have expected.

Acknowledgments

Part of this work was carried out during a visit of J L Jiménez to the University of Santander (Spain), supported by the Subdirección de Cooperación Internacional of the Ministerio de Educación y Ciencia (Spain), and UNAM (México).

Appendix 1

We want to calculate the expression

$$\boldsymbol{N}_E^{(s)} = -\frac{e^2}{c^2}\int \mathrm{d}^3 r \int \mathrm{d}^3 r' \frac{\rho(r)\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{r}\times\left[\dot{\boldsymbol{\omega}}\left(t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\times\boldsymbol{r}'\right]. \tag{A1.1}$$

Let us make the change $\boldsymbol{r}'-\boldsymbol{r}=\boldsymbol{R}$. With the notation $\boldsymbol{\Pi}(R)=\dot{\boldsymbol{\omega}}(t-R/c)$, we have

$$\boldsymbol{r}\times[\boldsymbol{\Pi}(R)\times(\boldsymbol{r}+\boldsymbol{R})]=r^2\left(I-\frac{\boldsymbol{rr}}{r^2}\right)\cdot\boldsymbol{\Pi}(R)+\boldsymbol{r}\cdot(\boldsymbol{R\Pi}-\boldsymbol{\Pi R}). \tag{A1.2}$$

The RHS of (A1.1) may be separated into two parts

$$\boldsymbol{N}_E^{(s)} = -(e^2/c^2)(\boldsymbol{N}_1+\boldsymbol{N}_2) \tag{A1.3}$$

with

$$\boldsymbol{N}_1=\int \mathrm{d}^3 r\int \mathrm{d}^3 R\frac{\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)}{R}\boldsymbol{r}\cdot(\boldsymbol{R\Pi}-\boldsymbol{\Pi R}), \tag{A1.4}$$

$$\boldsymbol{N}_2=\int \mathrm{d}^3 r\int \mathrm{d}^3 R\frac{\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)}{R}r^2\left(I-\frac{\boldsymbol{rr}}{r^2}\right)\cdot\boldsymbol{\Pi}(R). \tag{A1.5}$$

Let us first calculate N_1. Calling

$$\boldsymbol{\Gamma}(\boldsymbol{R})=\int \mathrm{d}^3 r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)\boldsymbol{r}, \tag{A1.6}$$

$\boldsymbol{N}_1$ can be written

$$\boldsymbol{N}_1=\int \mathrm{d}^3 R\, R^{-1}(\boldsymbol{\Pi R}-\boldsymbol{R\Pi})\cdot\boldsymbol{\Gamma}(\boldsymbol{R}). \tag{A1.7}$$

We are going to show that $\boldsymbol{\Gamma}(\boldsymbol{R})$ is a vector in the direction of $\boldsymbol{R}$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ form with $\boldsymbol{R}/R$ an orthonormal basis in $\mathbb{R}^3$. We shall prove that $\boldsymbol{\Gamma}\cdot\boldsymbol{u}=\boldsymbol{\Gamma}\cdot\boldsymbol{v}=0$. Let us write

$$\boldsymbol{\Gamma}\cdot\boldsymbol{u}=\int \mathrm{d}^3 r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)\boldsymbol{r}\cdot\boldsymbol{u}. \tag{A1.8}$$

Let us make the following change. If $\boldsymbol{r}=x_u\boldsymbol{u}+x_v\boldsymbol{v}+x_R\boldsymbol{R}/R$, let $\boldsymbol{r}'=-x_u\boldsymbol{u}+x_v\boldsymbol{v}+x_R\boldsymbol{R}/R$. Then

$$|\boldsymbol{r}|=|\boldsymbol{r}'| \qquad \text{and} \qquad |\boldsymbol{r}+\boldsymbol{R}|=|\boldsymbol{r}'+\boldsymbol{R}|.$$

Consequently

$$\boldsymbol{\Gamma}\cdot\boldsymbol{u}=\int \mathrm{d}^3 r'\,\rho(r')\rho(|\boldsymbol{r}'+\boldsymbol{R}|)(-)\boldsymbol{r}'\cdot\boldsymbol{u}=-\boldsymbol{\Gamma}\cdot\boldsymbol{u}=0$$

and the same for $\boldsymbol{\Gamma}\cdot\boldsymbol{v}$.

Therefore

$$\boldsymbol{\Gamma}(\boldsymbol{R})=(\boldsymbol{\Gamma}\cdot R/R^2)\boldsymbol{R}=\xi(R)\boldsymbol{R}/R \tag{A1.9}$$

$$\begin{aligned} \boldsymbol{N}_1 &= \int \mathrm{d}^3R\, R^{-1}\xi(R)\frac{(\boldsymbol{\Pi}R^2-\boldsymbol{RR}\cdot\boldsymbol{\Pi})}{R} \\ &= \int_0^\infty \mathrm{d}R\, R^2\xi(R)\boldsymbol{\Pi}(R)\cdot\int \mathrm{d}\Omega_R\left(I-\frac{\boldsymbol{RR}}{R^2}\right), \end{aligned} \tag{A1.10}$$

where $\mathrm{d}\Omega_R$ denotes the element of solid angle in the direction of $\boldsymbol{R}$.

Now, let

$$\vec{T}_1(R)=\int \mathrm{d}\Omega_R(I-\boldsymbol{RR}/R^2). \tag{A1.11}$$

If we make in this integral the following change

$$\boldsymbol{R}'=\mathcal{R}\boldsymbol{R}, \tag{A1.12}$$

where $\mathcal{R}$ denotes a rotation, it is easy to see that

$$\vec{T}_1(R)=\mathcal{R}^{-1}\vec{T}(R)\mathcal{R}, \tag{A1.13}$$

whence it follows that $\vec{T}$ is a multiple of the identity

$$\vec{T}_1\cdot\boldsymbol{\Pi}=\tfrac{1}{3}\operatorname{Tr}(\vec{T}_1)\boldsymbol{\Pi} \tag{A1.14}$$

and

$$\operatorname{Tr}(\vec{T}_1)=2\int \mathrm{d}\Omega_R=8\pi. \tag{A1.15}$$

Finally

$$\boldsymbol{N}_1=\tfrac{8}{3}\pi\int_0^\infty R^2\xi(R)\dot{\boldsymbol{\omega}}(t-R/c)\,\mathrm{d}R \tag{A1.16}$$

with

$$\xi(R)=\frac{\boldsymbol{\Gamma}\cdot\boldsymbol{R}}{R}=\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)\frac{\boldsymbol{r}\cdot\boldsymbol{R}}{R}. \tag{A1.17}$$

Let us consider now $\boldsymbol{N}_2$. Calling

$$\vec{T}_2(R)=\int \mathrm{d}\Omega_R\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)r^2\left(I-\frac{\boldsymbol{rr}}{r^2}\right), \tag{A1.18}$$

equation (A1.5) may be written

$$\boldsymbol{N}_2=\int_0^\infty \mathrm{d}R\, R\boldsymbol{\Pi}(R)\cdot\vec{T}_2(R). \tag{A1.19}$$

Now, an argument similar to the one developed for $\vec{T}_1(R)$ allows us to conclude that $\vec{T}_2(R)$ is also a multiple of the identity

$$\begin{aligned} \vec{T}_2(R)\boldsymbol{\Pi}(R) &= \tfrac{1}{3}\operatorname{Tr}(\vec{T}_2)\boldsymbol{\Pi}(R) \\ &= \tfrac{1}{3}\int \mathrm{d}\Omega_R\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)2r^2 \\ &= \tfrac{8}{3}\pi\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)r^2, \end{aligned} \tag{A1.20}$$

whence

$$\boldsymbol{N}_2=\tfrac{8}{3}\pi\int_0^\infty R\phi(R)\dot{\boldsymbol{\omega}}(t-R/c)\,\mathrm{d}R, \tag{A1.21}$$

where

$$\phi(R)=\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)r^2. \tag{A1.22}$$

Finally, taking into account (A1.16), (A1.17), (A1.21) and (A1.22), we obtain

$$\begin{aligned}\boldsymbol{N}_E^{(s)}&=-\frac{8\pi e^2}{3c^2}\int_0^\infty \mathrm{d}R[R^2\xi(R)+R\phi(R)]\dot{\boldsymbol{\omega}}\left(t-\frac{R}{c}\right)\\&=-\frac{8\pi e^2}{3c^2}\int_0^\infty \mathrm{d}R\,\dot{\boldsymbol{\omega}}\left(t-\frac{R}{c}\right)R\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)(\boldsymbol{r}\cdot\boldsymbol{R}+r^2).\end{aligned} \tag{A1.23}$$

Let now

$$G(t)=t\int \mathrm{d}^3r\,\rho(r)\rho(|\boldsymbol{r}+\boldsymbol{R}|)(\boldsymbol{r}\cdot\boldsymbol{R}+r^2)|_{R=ct}. \tag{A1.24}$$

Equation (A1.23) can be written

$$\begin{aligned}\boldsymbol{N}_E^{(s)}&=-\tfrac{8}{3}\pi e^2\int_0^\infty \mathrm{d}t'\,G(t')\dot{\boldsymbol{\omega}}(t-t')\\&=-\tfrac{8}{3}\pi e^2\int_{-\infty}^t \mathrm{d}t'\,G(t-t')\dot{\boldsymbol{\omega}}(t').\end{aligned}$$

Now, we only have to prove that this function G coincides with the one expressed in relation (29*b*). For this we substitute in equation (A1.24) $\rho(r)$ by expression (30)

$$G(t)=t\int \mathrm{d}^3r\int \mathrm{d}^3k\int \mathrm{d}^3k'(8\pi^3)^{-1}\tilde{\rho}(\boldsymbol{k})\tilde{\rho}(\boldsymbol{k}')\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\exp[\mathrm{i}\boldsymbol{k}'\cdot(\boldsymbol{r}+\boldsymbol{R})](\boldsymbol{r}\cdot\boldsymbol{R}+r^2). \tag{A1.25}$$

But

$$(\boldsymbol{r}\cdot\boldsymbol{R}+r^2)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}=(-\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\nabla}_k-\Delta_k)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{A1.26}$$

whence, making an integration by parts, and integrating over $\boldsymbol{r}$, we obtain

$$\begin{aligned}G(t)&=t\int \mathrm{d}^3k\,\mathrm{d}^3k'[(\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\nabla}_k-\Delta_k)\tilde{\rho}(\boldsymbol{k})]\tilde{\rho}(\boldsymbol{k}')\exp(\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{R})\delta^3(\boldsymbol{k}-\boldsymbol{k}')\\&=t\int \mathrm{d}^3k[(\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\nabla}_k-\Delta_k)\tilde{\rho}(\boldsymbol{k})]\tilde{\rho}(\boldsymbol{k})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}).\end{aligned} \tag{A1.27}$$

Now, in the term with the Laplacian, we again integrate by parts

$$\begin{aligned}\int \mathrm{d}^3k(\Delta_k\tilde{\rho})\tilde{\rho}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}}&=-\int \mathrm{d}^3k(\boldsymbol{\nabla}_k\tilde{\rho})\cdot\boldsymbol{\nabla}_k(\tilde{\rho}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}})\\&=-\int \mathrm{d}^3k(\tilde{\rho}')^2\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}}+\int \mathrm{d}^3k\,\mathrm{i}\boldsymbol{R}(\boldsymbol{\nabla}_k\tilde{\rho})\tilde{\rho}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}},\end{aligned} \tag{A1.27'}$$

where we have used that $\tilde{\rho}(\boldsymbol{k})=\tilde{\rho}(|\boldsymbol{k}|)=\tilde{\rho}(k)$. We then obtain

$$G(t)=t\int \mathrm{d}^3k[\tilde{\rho}'(k)]^2\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}}|_{R=ct}.$$

Integrating over solid angle take us finally to

$$G(t)=(4\pi/c)\int_0^\infty \mathrm{d}k\,k[\tilde{\rho}'(k)]^2\sin(kct), \tag{A1.28}$$

which is the required expression.

Appendix 2

We are going to calculate

$$\boldsymbol{N}_B^{(\mathrm{s})}=(e/c)\int \mathrm{d}^3r\,\rho(r)\boldsymbol{r}\cdot\boldsymbol{B}_{(\mathrm{s})}(\boldsymbol{r},t)(\boldsymbol{\omega}(t)\times\boldsymbol{r}), \tag{A2.1}$$

where $\boldsymbol{B}_{(\mathrm{s})}$ is given by

$$\boldsymbol{B}_{(\mathrm{s})}=\boldsymbol{\nabla}\times\frac{e}{c}\int\frac{\rho(r')[\boldsymbol{\omega}(t-|\boldsymbol{r}-\boldsymbol{r}'|/c)\times\boldsymbol{r}']}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3r'. \tag{A2.2}$$

Let us make the change $\boldsymbol{r}'-\boldsymbol{r}=\boldsymbol{R}$, and use equation (30) calling $\boldsymbol{\omega}(t-R/c)=\boldsymbol{\omega}_R$:

$$\begin{aligned}\boldsymbol{B}_{(\mathrm{s})}&=\frac{e}{c}\int \mathrm{d}^3R\frac{1}{R}\int \mathrm{d}^3k\frac{1}{(2\pi)^{3/2}}\tilde{\rho}(\boldsymbol{k})\boldsymbol{\nabla}_r\times[\boldsymbol{\omega}_R\times(\boldsymbol{r}+\boldsymbol{R})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}+\boldsymbol{R})}]\\&=\frac{e}{c(2\pi)^{3/2}}\int \mathrm{d}^3R\frac{1}{R}\int \mathrm{d}^3k\,\tilde{\rho}(k)[\boldsymbol{\nabla}_k\cdot\boldsymbol{k}\boldsymbol{\omega}_R-\boldsymbol{\nabla}_k(\boldsymbol{\omega}_R\cdot\boldsymbol{k})]\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}+\boldsymbol{R})}.\end{aligned} \tag{A2.3}$$

where the gradients act over everything on their right. Integrating by parts, we have

$$\boldsymbol{B}_{(\mathrm{s})}=-\frac{e}{c(2\pi)^{3/2}}\int \mathrm{d}^3R\frac{1}{R}\int \mathrm{d}^3k\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}+\boldsymbol{R})}\tilde{\rho}'(k)k\left(I-\frac{\boldsymbol{kk}}{k^2}\right)\cdot\boldsymbol{\omega}_R. \tag{A2.4}$$

Substituting (A2.4) into (A2.1), and using (30), we obtain

$$\boldsymbol{N}_B^{(\mathrm{s})}=-\frac{e^2}{8\pi^3c^2}\boldsymbol{\omega}(t)\times\int_0^\infty R\vec{T}_3(R)\cdot\boldsymbol{\omega}_R\,\mathrm{d}R \tag{A2.5}$$

where

$$\vec{T}_3(R)=\int \mathrm{d}\Omega_R\int \mathrm{d}^3k\int \mathrm{d}^3k'\int \mathrm{d}^3r\,\tilde{\rho}(k')\tilde{\rho}'(k)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}+\boldsymbol{R})}[k^2\boldsymbol{rr}-(\boldsymbol{r}\cdot\boldsymbol{k})\boldsymbol{rk}]k^{-1}. \tag{A2.6}$$

Again, it is easy to see that $\vec{T}_3(R)$ is a multiple of the identity. Then

$$\vec{T}_3(R)=\tfrac{1}{3}\,\mathrm{Tr}[\vec{T}_3(R)],$$

whence

$$\begin{aligned}\boldsymbol{N}_B^{(\mathrm{s})}=-\frac{e^2}{8\pi^3c^2}\boldsymbol{\omega}(t)\times\tfrac{1}{3}\int_0^\infty R\,\mathrm{d}R\int \mathrm{d}\Omega_R\int \mathrm{d}^3k\int \mathrm{d}^3k'\int \mathrm{d}^3r\,\tilde{\rho}(k')\\\times\tilde{\rho}'(k)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}+\boldsymbol{R})}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}}k^{-1}[k^2r^2-(\boldsymbol{k}\cdot\boldsymbol{r})^2]\boldsymbol{\omega}_R.\end{aligned} \tag{A2.7}$$

Introducing the differential operator $D_{k'}(k)$

$$D_{k'}(k)\varphi(k') = [-k^2\nabla_{k'}\cdot\nabla_{k'} + (\boldsymbol{k}\cdot\nabla_{k'})^2]\varphi(k') \tag{A2.8}$$

and making an integration by parts over k', equation (A2.7) may be written as follows:

$$\begin{aligned}\boldsymbol{N}_B^{(s)} = -\frac{e^2}{8\pi^3c^2}\boldsymbol{\omega}(t)\times\tfrac{1}{3}\int_0^\infty R\,\mathrm{d}R\int\mathrm{d}\Omega_R\int\mathrm{d}^3k\int\mathrm{d}^3k'\int\mathrm{d}^3r\\ \times[D_{k'}(k)\tilde{\rho}(k')]\tilde{\rho}'(k)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}+\boldsymbol{R})}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}}K^{-1}\boldsymbol{\omega}_R,\end{aligned} \tag{A2.9}$$

from which, integrating over r and k', it results that

$$\boldsymbol{N}_B^{(s)} = -\frac{e^2}{3c^2}\boldsymbol{\omega}(t)\times\int_0^\infty R\,\mathrm{d}R\int\mathrm{d}\Omega_R\int\mathrm{d}^3k\,k^{-1}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}}\tilde{\rho}'(k)[D_{k'}(k)\tilde{\rho}(k')]_{k'=k}\boldsymbol{\omega}_R. \tag{A2.10}$$

A straightforward calculation gives

$$D_{k'}(k)\tilde{\rho}(k')|_{k'=k} = -2k\tilde{\rho}'(k),$$

whence

$$\boldsymbol{N}_B^{(s)} = \frac{2e^2}{3c^2}\boldsymbol{\omega}(t)\times\int_0^\infty R\,\mathrm{d}R\,\boldsymbol{\omega}_R\int\mathrm{d}\Omega_R\int\mathrm{d}^3k[\tilde{\rho}'(k)]^2\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}} \tag{A2.11}$$

and integrating over the solid angle Ω_R, we finally obtain expression (28*b*).

Appendix 3

We want to prove equation (53). Substituting the expression (54) and using

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}\times\boldsymbol{c}) = a^2(\boldsymbol{b}\cdot\boldsymbol{c}) - (\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\cdot\boldsymbol{c})$$

we obtain

$$\mathscr{F} = \boldsymbol{\omega}(t)\overleftrightarrow{T}_4\cdot\boldsymbol{\omega}(t)e^2/2c^2 \tag{A3.1}$$

where

$$\overleftrightarrow{T}_4 = \int\mathrm{d}^3r\int\mathrm{d}^3r'\,\frac{\rho(r)\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}(\boldsymbol{r}\cdot\boldsymbol{r}' - \boldsymbol{r}\boldsymbol{r}'). \tag{A3.2}$$

Again, $\overleftrightarrow{T}_4$ is a multiple of the identity and then

$$\overleftrightarrow{T}_4 = \tfrac{1}{3}\,\mathrm{Tr}(\overleftrightarrow{T}_4) = \tfrac{1}{3}\int\mathrm{d}^3r\int\mathrm{d}^3r'\,\frac{\rho(r)\rho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}2(\boldsymbol{r}\cdot\boldsymbol{r}') \tag{A3.3}$$

which gives equation (53).

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